Int J. Heat Mass Transfer. Vol. 9, pp. 987–992. Pergamon Press 1966. Printed in Great Britain

SHORTER COMMUNICATION

APPLICATION OF CASE'S METHOD TO PLANE PARALLEL RADIATIVE TRANSFER*

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(*Received* 5 *February* 1965 *and in revised form* 25 *October* 1965)

1. INTRODUCTION

THIS NOTE is intended to illustrate the application to radiative transfer of a method which has enjoyed great success in the related area of neutron transport theory. Basically, the method involves the construction of a complete set of eigenfunctions of the homogeneous transport equation in terms of which any other solution may be expanded. Sources and boundary conditions determine the coefficients in the expansion. This method, originally developed by Case [1], is quite analogous to the classical approach used in heat conduction problems. The only major difference is that the "eigenfunctions" are generalized functions in the sense of Schwartz [2], e.g. they are not square integrable in the ordinary sense.

Here we show how the method may be applied to the simple case of radiative transfer in grey gases (or, equivalently, in a gas with purely isotropic scattering) in plane geometry. As an example, we solve the problem of heat transfer between parallel plates at arbitrary temperatures and with arbitrary emissivities. As a result of this analysis one obtains very accurate approximate formulae for the heat transfer and temperature distribution; these are of the same form as those obtained by a different method by Heaslet and Warming [3]. From these results one is also able to infer how the boundary conditions in the diffusion approximation might be modified to give improved results in other geometries.

Usiskin and Sparrow [4] obtained one of the first solutions of the parallel plate probiem by numerical methods. Other authors have since considered specific aspects of this problem. (Heaslet and Warming [3] reference many of these papers.) For example, Olfe and Penner [5}, and Probstein [6] have concerned themselves with improving the boundary conditions in the diffusion approximation.

* Work supported in part by the National Science Foundation Grant GP-1522.

† National Science Foundation Pre-doctoral Fellow.

2. REDUCTION TO THE ALBEDO PROBLEM

The first problem to be considered is posed as follows: Determine the radiation intensity $I(x, \mu)$ in a region containing a grey absorbing (and emitting) gas bounded by two black plates at temperatures T_1 and T_2.

For the case of a grey gas in radiative and local thermodynamic equilibrium the steady state one-dimensional equation of transfer is [7, 8]:

$$\mu \frac{\partial I}{\partial x}(x, \mu) + I(x, \mu) = \tfrac{1}{2}\int_{-1}^{1} I(x, \mu')\, d\mu' \tag{1}$$

where x is a dimensionless variable ($\varkappa z$) and $I(x, \mu)$ is the integrated (over frequency) intensity. This equation also applies to the case of a non-absorbing, non-emitting gas with perfect isotropic scattering [8] (or a combination of grey absorption and isotropic scattering).

For the case of black boundaries, the boundary conditions to be applied to equation (1) are

$$\begin{aligned} I\left(-\frac{d}{2}, \mu\right) &= \frac{\sigma}{\pi} T_1^4 = f_1 \qquad (\mu > 0) \\ I\left(\frac{d}{2}, \mu\right) &= \frac{\sigma}{\pi} T_2^4 = f_2 \qquad (\mu < 0) \end{aligned} \tag{2}$$

where d is the dimensionless distance between the plates. These equations define f_1 and f_2. In order to be entirely general (and for later application) we will allow f_1 and f_2 to be functions of μ, i.e. we will allow the surfaces to be non-diffuse.

It is sufficient to consider the problem in which $f_2 = 0$ and $f_1 = f(\mu)$ since the solution of the original problem can be constructed from the solutions $I_1(x, \mu)$ and $I_2(x, \mu)$ corresponding to $f(\mu) = f_1(\mu)$ and $f(\mu) = f_2(-\mu)$:

$$I(x, \mu) = I_1(x, \mu) + I_2(-x, -\mu). \tag{3}$$

This problem is essentially the slab albedo problem of one-speed neutron transport theory which was considered by McCormick and Mendelson [9].

3. USE OF CASE'S METHOD

Case [1] developed a method for treating one-dimensional neutron transport problems which uses an expansion in the (singular) eigenfunctions of the Boltzmann equation. He proved that the set of eigenfunctions of the Boltzmann equation that he derived is complete and orthogonal for $\mu\epsilon(-1, 1)$, and that a truncated set is complete for $\mu\epsilon(0, 1)$.

Since the Boltzmann equation, (1), is precisely the one-speed neutron transport equation which Case considered, Case's method is applicable to the present problem. Thus we can always expand $I(x, \mu)$ in terms of Case's eigenfunctions

$$I(x, \mu) = a + b(x - \mu) + \int_{-1}^{1} A(k)\, \varphi_k(\mu)\, e^{-x/k}\, dk \tag{4}$$

where $\varphi_k(\mu)$ is the continuum eigenfunction of Case:

$$\varphi_k(\mu) = \tfrac{1}{2}P\frac{k}{k-\mu} + \lambda(k)\,\delta(\mu - k) \tag{5}$$

and

$$\lambda(k) = 1 - k\tanh^{-1} k. \tag{6}$$

The P indicates that singular integrals are to be evaluated in the Cauchy principal value sense. The problem is to choose a, b and $A(k)$ in such a way that the boundary conditions are satisfied exactly.

If we apply the boundary conditions to the expansion (4), follow McCormick and Mendelson's [9] method for adding and subtracting the two equations, use the half-range orthogonality relations of Kuscer, McCormick, and Summerfield [10] and a result of Shure and Natelson [11] then we obtain the following equations for the expansion coefficients:

$$a = \tfrac{1}{2}\int_0^1 \gamma(\mu) f(\mu)\, d\mu - \tfrac{1}{4}\int_0^1 k\, B_+(k)\, e^{-d/k}\, X(-k)\, dk \tag{7}$$

$$b = \frac{-1}{d + 2z_0}\left[\int_0^1 \gamma(\mu) f(\mu)\, d\mu + \tfrac{1}{2}\int_0^1 k\, B_-(k)\, e^{-d/k}\, X(-k)\, dk\right] \tag{8}$$

$$\int_0^1 \gamma(\mu)\, \varphi_{k'}(\mu) f(\mu)\, d\mu = \tfrac{3}{2}\frac{k'}{g(1, k')\, X(-k')} B_+(k') + + \frac{k'}{4}\int_0^1 B_+(k)\, e^{-d/k}\frac{k}{k + k'} X(-k)\, dk \tag{9}$$

$$\int_0^1 \gamma(\mu)\, \varphi_{k'}(\mu) f(\mu)\, d\mu = bk' + \tfrac{3}{2}\frac{k'}{g(1, k')\, X(-k')} B_-(k') - - \frac{k'}{4}\int_0^1 B_-(k)\, e^{-d/k}\frac{k}{k + k'} X(-k)\, dk \tag{10}$$

where [9]

$$B_{\pm}(k) = [A(k) \pm A(-k)]\, e^{d/2k}, \qquad \gamma(\mu) = \tfrac{3}{2}\frac{\mu}{X(-\mu)} \tag{11}$$

and

$$X(z) = \frac{1}{1 - z}\exp\left[\frac{1}{2\pi i}\int_0^1 \ln\left(\frac{\lambda(k) + i\pi k/2}{\lambda(k) - i\pi k/2}\right)\frac{dk}{k - z}\right] \tag{12}$$

and z_0 is the Milne problem extrapolation length (0·710446) [11, 12].

Equations (9) and (10) are Fredholm (ordinary) integral equations for the coefficients $B_{\pm}(k)$ while equations (7) and (8) are side conditions that determine a and b. At first sight these equations look very complicated but we will show that the integral terms are small and can be neglected for most applications. The expansion coefficients can be determined from (7)–(10) to any degree of accuracy desired.

The use of these equations is best illustrated by an example for which the solution is well known. We will consider the diffuse case, i.e. $f_1(\mu) = f =$ a constant. With no loss of generality we choose $f = 1$. The quantities associated with this problem will be denoted by a zero subscript, e.g. $I_0(x, \mu)$.

For this case it can be shown that $B_{0+} = 0$ and $a_0 = \frac{1}{2}$.

We may now obtain the coefficients b_0 and $B_{0-}(k)$ from equations (8) and (10). Although numerical solution is not difficult, the smallness of the integral terms in these equations leads to a rapidly convergent (and simple) iterative solution. A first approximation is to neglect the integrals entirely, i.e.

$$b_0^{(1)} = \frac{-1}{d + 2z_0} \tag{13}$$

$$B_0^{(1)}(k') = -\tfrac{2}{3}b_0^{(1)}\, g(1, k')\, X(-k'). \tag{14}$$

The second approximation is obtained by the usual iteration procedure for equations of this type.

We now define

$$\rho(x) = 2\pi\int_{-1}^{1} I(x, \mu)\, d\mu = 4\sigma T_g^4(x)$$

$$= 2\pi\left[2a + 2bx + \int_{-1}^{1} A(k)\, e^{-x/k}\, dk\right] \tag{15}$$

where T_g is the local temperature of the gas, and we have used the expansion (4). The $\rho(x)$ belonging to the particular problem treated above ($f_1 = 1, f_2 = 0$) will be designated $\rho_0(x)$.

Now, consider the problem of two black plates with radiation intensities f_1 and f_2, where $f_i = \sigma/\pi T_i^4$ ($f_1 > f_2$). The solution to this problem is readily seen to be:

$$\rho(x) = 4\pi f_2 + (f_1 - f_2)\,\rho_0(x). \tag{16}$$

The heat transfer is easily computed. By definition:

$$q = 2\pi \int_{-1}^{1} \mu I(x, \mu)\,d\mu = -\frac{4\pi}{3} b = -\frac{4\sigma}{3} b_0(T_1^4 - T_2^4). \quad (17)$$

Using the first approximation for b_0 we obtain

$$q^{(1)} = \tfrac{4}{3}\frac{\sigma}{(d + 2z_0)}(T_1^4 - T_2^4). \quad (18)$$

This approximation is sufficiently accurate for many engineering purposes and in particular for large optical thicknesses. The maximum error occurs at $d = 0$ for which equation (18) predicts $q^{(1)} = 0{\cdot}938\,\sigma(T_1^4 - T_2^4)$ which is only 6 per cent below the exact result. Numerical results are treated in greater detail in Section 6.

4. EXTENSIONS

We now consider the case in which the walls both emit and reflect. The total energy leaving either plate is called the radiosity and includes both the emitted and the reflected radiation. The radiosities are

$$R_1(\mu) = \epsilon_1 f_1(\mu) + 2\int_{-1}^{0} |\mu'|\, r_1(\mu' \to \mu)\, I(-d/2, \mu')\,d\mu' \quad (\mu > 0) \quad (19)$$

$$R_2(\mu) = \epsilon_2 f_2(\mu) + 2\int_{0}^{1} |\mu'|\, r_2(\mu' \to \mu)\, I(d/2, \mu')\,d\mu' \quad (\mu < 0) \quad (20)$$

where $r(\mu' \to \mu)$ is the angular dependent reflectivity and ϵ is the emissivity.

For purposes of illustration we consider diffuse reflection, $[r(\mu' \to \mu) = r = 1 - \epsilon]$ and assume that f_1 and f_2 are independent of μ. In this case both R_1 and R_2 are independent of μ (on their respective ranges) so we have, immediately:

$$I(x, \mu) = R_1 I_0(x, \mu) + R_2 I_0(-x, -\mu). \quad (21)$$

Substituting (21) into equations (19) and (20) and performing the required integrations gives two algebraic equations for the two unknowns R_1 and R_2. These equations are readily solved. We write down only the result for the heat transfer:

$$q = \frac{-4\sigma}{3} b_0 \left[\frac{T_1^4 - T_2^4}{\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1\right) - 2K\left[\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 2\right]}\right] \quad (22)$$

where

$$K = \int_0^1 \mu I_0(-d/2, -\mu)\,d\mu = \tfrac{1}{4}[1 - b_0(d - \tfrac{4}{3})] + \int_{-1}^{1} A(k)\, e^{+d/2k} \int_0^1 \mu \varphi_{-k}(\mu)\,d\mu\,dk. \quad (23)$$

We note that no approximations have been made in the derivation of (22). Equation (22) reduces to equation (18) when $\epsilon_1 = \epsilon_2 = 1$. Furthermore, in the limit, $d = 0$, $b_0 = -\frac{3}{4}$ and the integral term in equation (23) vanishes so that $K = 0$ and equation (22) reduces to a well-known result for non-black parallel plates separated by a transparent gas.

Neglecting the integral over $A(k)$ in equation (23), and using the first approximation for b_0 we obtain:

$$q^{(1)} = \frac{4\sigma}{3(d + 2z_0)}\left[\frac{T_1^4 - T_2^4}{\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1\right) - \frac{1}{2}\left[1 + \frac{d - \frac{4}{3}}{d + 2z_0}\right]\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 2\right)}\right] \quad (24)$$

which is of the form obtained by Heaslet and Fuller [13]. This result is sufficiently accurate for engineering purposes; numerical values are given in Section 6.

A similar procedure can be used for the more general boundary conditions, in which

$$r(\mu' \to \mu) = \sum_{n=1}^{N} p^{(n)}(\mu')\, q^{(n)}(\mu). \quad (25)$$

For $N = 1$ this leads to four separate problems for which the solutions are given by equations (7)–(10). Two of these solutions contain arbitrary constants which represent the integrals in (19) and (20). Substituting the solutions into the definitions of these constants yield two algebraic equations for the two unknown constants. Similarly, in the general case one gets a set of $2N$ algebraic equations for $2N$ unknown constants.

Specular (mirror) reflection requires separate treatment but is no more difficult than the problem which has already been solved; however, it will not be treated here.

As explained above, the solutions obtained are applicable to the case of isotropic scattering. For problems involving anisotropic scattering the solution may be found by a combination of the procedure used by Mika [14] and that presented here. A method of extending this procedure to time-dependent problems is given by Case [6].

5. THE DIFFUSION APPROXIMATION

The diffusion approximation to the transport equation (1) and its solution in plane geometry are

$$\nabla^2 \rho(x) = 0, \qquad \rho(x) = a + bx. \quad (26)$$

Table 1

k	$X(-k)$	$\gamma(k)$	$g(l,k)$
0·00	1·7321	0·0000	1·0000
0·05	1·5274	0·0491	0·9988
0·10	1·3932	0·1077	0·9953
0·15	1·2873	0·1748	0·9894
0·20	1·1994	0·2501	0·9811
0·25	1·1244	0·3335	0·9703
0·30	1·0594	0·4248	0·9570
0·35	1·0021	0·5239	0·9409
0·40	0·9513	0·6307	0·9220
0·45	0·9056	0·7453	0·9001
0·50	0·8644	0·8676	0·8749
0·55	0·8271	0·9975	0·8461
0·60	0·7927	1·1351	0·8134
0·65	0·7616	1·2802	0·7761
0·70	0·7327	1·4330	0·7335
0·75	0·7061	1·5933	0·6845
0·80	0·6813	1·7613	0·6274
0·85	0·6583	1·9368	0·5595
0·90	0·6368	2·1198	0·4752
0·95	0·6168	2·3105	0·3604
1·00	0·5979	2·5086	0·0000

Within the framework of diffusion theory the heat fluxes to the right (+) and left (−) are given by [15]:

$$q_{\pm}(x) = \frac{\rho(x)}{4} \mp \tfrac{1}{6}\nabla\rho(x). \tag{27}$$

For the problem treated in Section 4 the diffusion theory boundary conditions are

$$\begin{aligned} q_+(-d/2) &= r_1 q_-(-d/2) + \pi\epsilon_1 f_1 \\ q_-(d/2) &= r_2 q_+(d/2) + \pi\epsilon_2 f_2. \end{aligned} \tag{28}$$

Using equations (26) and (27), one easily obtains the coefficients a and b.

The net heat transfer is simply

$$q = q_+ - q_- = -\tfrac{1}{3}\nabla\rho = -b/3. \tag{29}$$

Substituting the value of b obtained by the above calculation, one has

$$q = \tfrac{4}{3}\frac{\sigma(T_1^4 - T_2^4)}{d + \tfrac{4}{3}\left[\dfrac{1}{\epsilon_1} + \dfrac{1}{\epsilon_2} - 1\right]}. \tag{30}$$

This is the equation (58a) of Heaslet and Warming [3], which we conclude must correspond to the diffusion approximation. We note that the solution indicates that the boundary conditions for the diffusion approximation could as well be formulated in terms of extrapolated boundaries as is done in nuclear reactor theory (i.e. that $\rho = 4\sigma T^4$ at a distance $\frac{2}{3}$ from the boundary). Such a formulation is equivalent to the use of "slip" boundary conditions. We note that the first approximation (24) reduces to (30) if $2z_0(= 1·4208)$ is replaced by $\frac{4}{3}(= 1·3333)$. Thus with the given boundary conditions, diffusion theory predicts the heat flow quite adequately. Essentially, diffusion theory gives the discrete terms in $\rho(x)$ [i.e. the first two terms in equation (15)] fairly well but it fails to predict the continuum (integral) term at all; thus it does not give the temperature distribution as accurately as the approximations given earlier. Numerical evidence for these conclusions is given in the next section.

Since diffusion theory gives reasonably accurate results, and is extremely simple to use (note that equation (26) is just the steady state conduction equation) one would like to apply it to other geometries. The only difficulty is that due to curvature effects the boundary conditions (28) with equations (27) for $q_{\pm}$ are known to give poor results for

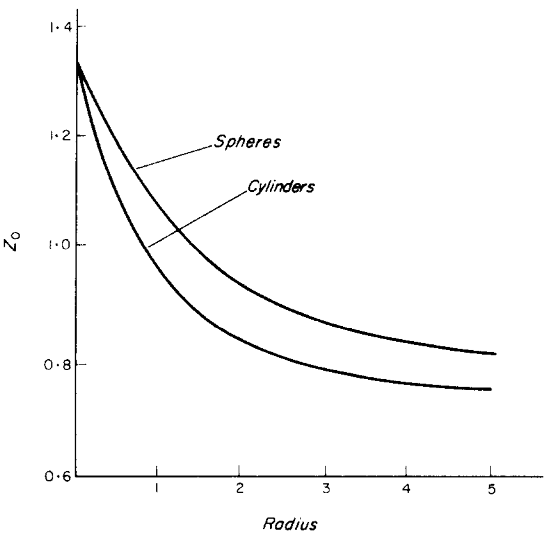

FIG. 1. Dimensionless extrapolation length z_0 as a function of radius for spheres and cylinders (from Davison [16]).

other (e.g. cylindrical) geometries To some extent this can be corrected as follows.

We note that diffusion theory would yield equation (18) for the heat transfer if in the boundary conditions the term $\nabla\rho/6$ were replaced by $z_0\nabla\rho/4$. For other geometries it is reasonable to use these modified boundary conditions with z_0 a function of the curvature of the boundaries. (This approximation is known to give good results in similar problems in neutron transport theory.) Approximate values of z_0 for spheres and cylinders were given by Davison [16] and are included here as Fig. 1. Using this modified diffusion theory the radiation problems can be solved to good approximation with the use of existing computer codes for conduction problems.

6. NUMERICAL RESULTS

The results of some sample calculations are now given in order to indicate the accuracy of the various approximations. All errors are relative to exact solutions obtained by numerical methods. These are in precise agreement with the solutions of other authors.

In Fig. 2 the relative errors in $\rho_0(x)$ are given for a rather unfavorable case ($d = 0{\cdot}1$). The first approximation [equations (13) and (14)] yields less than one per cent error whereas the second approximation is essentially exact (maximum error: 0·03 per cent). The diffusion theory results are worse than either of these approximations but are still quite good. It is to be noted that the error in T^4 will be less than the error in ρ_0 when both plates are at finite temperatures.

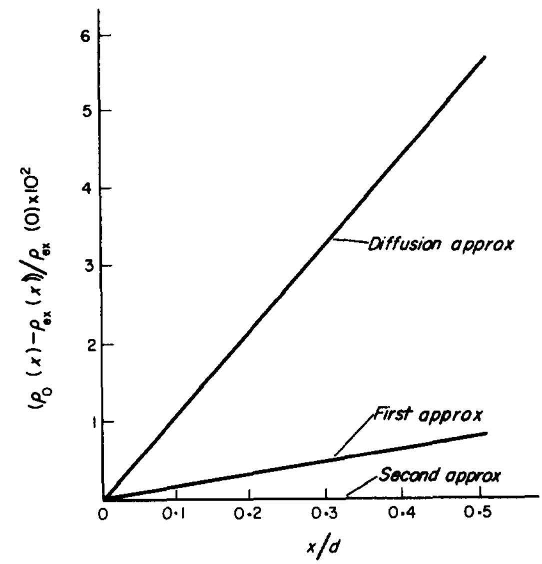

FIG. 2. Relative error in $\rho_0(x)$ in various approximations for $d = 0{\cdot}1$. Note that $x = 0$ corresponds to the center line and that the error at $-x$ is the negative of the error at x.

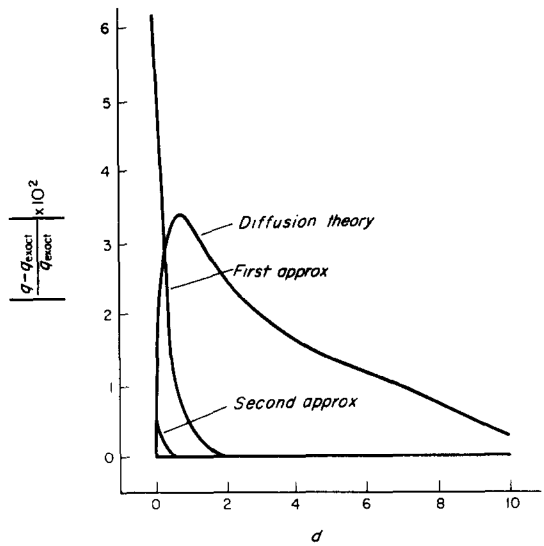

FIG. 3. Relative error in q for the case of two black walls. The first approximation is always higher than the exact result; diffusion theory is always lower.

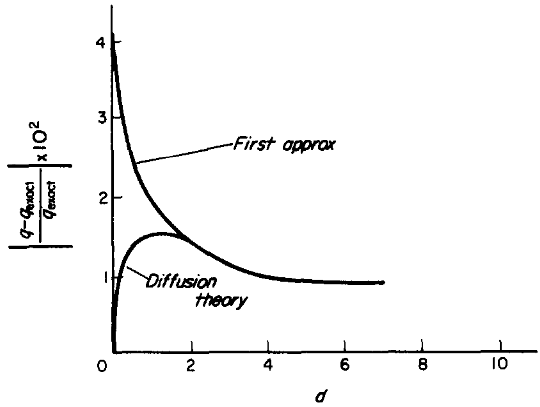

FIG. 4. Relative error in q for diffusely reflecting walls ($\epsilon_1 = \epsilon_2 = 0{\cdot}5$). The first approximation is always lower than the exact result; diffusion theory is always high.

The error in the heat transfer rate in various approximations for the case of two black walls is given in Fig. 3. In the first approximation the maximum error is 6·2 per cent; this is reduced by a factor of ten in the second approximation. Diffusion theory accidentally gives the correct value at $d = 0$ but is worse than the first approximation for $d > 0{\cdot}5$.

In Fig. 4 similar results are given for the case of partially reflecting walls. These results are typical of a large number of cases that were run. Diffusion theory [equation (30)] is always exact in the limit $d = 0$ and is always superior to the first approximation, equation (24), up to about $d = 1$. For $d > 1$ the two methods give nearly equivalent results with equation (24) generally being slightly better. For the temperature distribution, the results obtained by approximating the integral equation are always superior.

In conclusion, we have presented a method of arriving at accurate, but simple, results for radiative transfer problems under a wide range of boundary conditions. It has also been shown that, with the proper boundary conditions, diffusion theory can be made to yield results of sufficient accuracy for most engineering purposes.

ACKNOWLEDGEMENTS

We would like to thank Professor George Leppert for several helpful discussions and the Stanford Computation Center for use of the computer in compiling the numerical results.

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